denote the coordinate of the object at time $t$. The *velocity* (or "instantaneous velocity") of the object at time $t$ is:

$$v(t) = s'(t).$$

The *acceleration* of the object at time $t$ is:

$$a(t) = v'(t) = s''(t).$$

**Notation** Let $y = f(x)$.

| | |
|---|---|
| 1st derivative of $f$: | $\frac{dy}{dx} = \frac{df}{dx} = f'(x)$ |
| 2nd derivative of $f$: | $\frac{d^2y}{dx^2} = \frac{d^2f}{dx^2} = f''(x)$ |
| $\vdots$ | $\vdots$ |
| $n$-th derivative of $f$: | $\frac{d^ny}{dx^n} = \frac{d^nf}{dx^n} = f^{(n)}(x)$ |

**Example 5.2.1.**

1. $$\frac{d^n}{dx^n}(e^x) = e^x, \quad \frac{d^n}{dx^n}(a^x) = a^x \cdot (\ln a)^n.$$

2. $y = x^n, n \in \mathbb{N}$.

$y' = n x^{n-1}$

$y'' = n(n-1)x^{n-2}$

$$y^{(m)} = \begin{cases} n(n-1)(n-2)\cdots(n-m+1)x^{n-m}, & \text{if } m < n, \\ n(n-1)(n-2)\cdots 2 \cdot 1 = n!, & \text{if } m = n, \\ 0, & \text{if } m > n. \end{cases}$$

$y^{(n)} = n!$

$y^{(n+1)} = 0$

**Example 5.2.2.** Let $y$ be defined implicitly by the equation $x^2 + y^2 + e^{xy} = 2$. Find $y'$ and $y''$ at $x = 1$. → $y = 0$

$\frac{d}{dx}$ $\frac{d}{dx} = 0$

$u = xy$

*Solution.* Differentiate both sides of the preceding equation with respect to $x$ to get

$\frac{d}{dx}(e^{xy}) = \frac{d}{dx}e^u = \frac{d}{du}e^u \frac{du}{dx} = e^u(x'y + xy')$

plug in $x=1$ $y=0$ → $$\frac{d}{dx}\left(2x + 2yy' + e^{xy}(y + xy')\right) = \frac{d}{dx}0. \quad ----(1)$$

$2 + 1\cdot(y' = 0$

$y'|_{x=1} = -2$

Then differentiate both sides of the equation with respect to $x$ one more time to get

plug in $x=1$, $y=0$ → $$2 + 2y'y' + 2yy'' + e^{xy}(y + xy')^2 + e^{xy}(2y' + xy'') = 0. \quad ----(2)$$

$y'|_{x=1} = -2$

$2 + (4) + 1\cdot(1\cdot(-2))^2 + 1\cdot(2\cdot(-2) + y'') = 0$

$\frac{d}{dx}(e^{xy}(y + xy')) = \left(\frac{d}{dx}e^{xy}\right)(y + xy') + e^{xy}(y' + y' + xy'') = e^{xy}(y + xy')^2 + e^{xy}(2y' + xy'')$

$\frac{d}{dx}(yy') = y'y' + yy''$

Inserting $x = 1, y = 0$ into Equations (1), (2), we have:

$$y'|_{x=1} = -2,$$
$$y''|_{x=1} = -10.$$

■

**Example 5.2.3.** Suppose that $y = e^{\lambda x}$ satisfies $y'' - 2y' - 3y = 0$ (a "differential equation"). Find the constant $\lambda$.

$\frac{d}{dx} e^{\lambda x} = \frac{d}{du} e^u \frac{du}{dx} = e^{\lambda x} \lambda \qquad u = \lambda x$

*Solution.* $y = e^{\lambda x}$ implies that $y' = \lambda e^{\lambda x}$, which in turn implies $y'' = \lambda^2 e^{\lambda x}$.

Combining the preceding identities with the equation $y'' - 2y' - 3y = 0$, we have:

$$(\lambda^2 - 2\lambda - 3)e^{\lambda x} = 0.$$

Since $e^{\lambda x} \neq 0$ for all $x$,

$$\lambda^2 - 2\lambda - 3 = 0, \rightarrow \lambda = -1, 3.$$

■

$\frac{d^n}{dx^n}(e^{\lambda x}) = \lambda^n e^{\lambda x}$

More generally, if $y = e^{\lambda x}$ solves

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0,$$

then

$$(a_n \lambda^{(n)} + a_{n-1} \lambda^{(n-1)} + \cdots + a_1 \lambda + a_0) e^{\lambda x} = 0,$$

$\Rightarrow$

$$a_n \lambda^{(n)} + a_{n-1} \lambda^{(n-1)} + \cdots + a_1 \lambda + a_0 = 0.$$

**Exercise 5.2.1.** Find constants $\lambda$ such that $y = e^{\lambda} x$ satisfies $y''' - 2y'' - 3y' = 0$.
Answer: $\lambda = -1, 0, 3$.

$(\lambda^3 - 2\lambda^2 - 3\lambda) = 0$

$= \lambda(\lambda^2 - 2\lambda - 3)$

$= \lambda(\lambda - 3)(\lambda + 1) = 0$

$\Rightarrow \lambda = 0, 3, -1.$

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Chapter 6: Application of Derivatives I

**Learning Objectives**:
(1) Apply L'Hôpital's rule to find limits of indeterminate forms.
(2) Discuss increasing and decreasing functions.
(3) Define critical points and relative/absolute extrema of real functions of 1 variable.
(4) Use the first derivative test to study relative/absolute extrema of functions.

## 6.1 Limits of indeterminate forms and L'Hôpital's rule

Recall the Remark in the end of Section 2.4 regarding exceptional cases of limits, which can not be computed using the algebraic rules of limits in Proposition 2, but the limits might still exist. Limits of this type are said to be of indeterminate forms.

### 6.1.1 Limits of indeterminate forms $\frac{0}{0}, \frac{\infty}{\infty}$

Consider $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$,

1. if $\lim\limits_{x\to a} f(x) = A$, $\lim\limits_{x\to b} g(x) = B \neq 0$, $A, B \in \mathbb{R}$, then by the quotient rule,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \frac{A}{B}.$$

2. if $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0 \quad (\pm\infty)$, then the quotient rule is not applicable. Limits of this type are said to be of indeterminate form type $\frac{0}{0}$ or type $\frac{\infty}{\infty}$

   For example,

$$\lim_{x\to 1}\frac{x^2-1}{x^3-1}, \quad \left(\text{type } \frac{0}{0}\right)$$

$$\lim_{x\to+\infty}\frac{x+1}{2x+3}, \quad \lim_{x\to+\infty}\frac{-x+1}{2x^3}, \quad \left(\text{type } \frac{\infty}{\infty}\right).$$

**Theorem 6.1.1** (L'Hôpital's rule for limits of types $\frac{0}{0}, \frac{\infty}{\infty}$).

*Let $f(x)$, $g(x)$ be differentiable and suppose that $g'(x) \neq 0$ near the point $a$.*

*If*

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \quad \text{or} \quad \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty,$$

*then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

*Remark.* (a) An intuitive explanation: When $f(a) \approx 0 \approx g(a)$,

$$\frac{f(x)}{g(x)} \approx \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}.$$

(b) The statement of the theorem still holds if "$x \to a$" is replaced by "$x \to \pm\infty$" or "$x \to a^{\pm}$". It also holds if $\lim_{x\to a} f(x) = \pm\infty$ $\lim_{x\to a} g(x) = \mp\infty$. (Use $\lim_{x\to a} \frac{f(x)}{g(x)} = -\lim_{x\to a} \frac{-f(x)}{g(x)}$ and apply the theorem to $\lim_{x\to a} \frac{-f(x)}{g(x)}$.)

**Example 6.1.1.** Limits of type $\frac{0}{0}$

1.

$$\lim_{x\to 1} \frac{x^2-1}{x^3-1} \qquad \text{(check condition 1: } \tfrac{0}{0}\text{)}$$
$$= \lim_{x\to 1} \frac{2x}{3x^2} \qquad \text{(check condition 2: this limit is } \tfrac{2}{3}\text{)}$$
$$= \frac{2}{3}. \qquad = \lim_{x\to 1} \frac{2}{3x}$$

*Remark.* Alternatively, use the "canceling common factors" trick in the previous chapters.

2.

$$\lim_{x\to 1} \frac{e^x - e}{\sqrt{x} - 1} \qquad \text{(the limit is of type } \tfrac{0}{0}\text{)}$$
$$= \lim_{x\to 1} \frac{e^x}{\frac{1}{2}x^{-\frac{1}{2}}} \underset{\text{quotient rule}}{=} \frac{\lim_{x\to 1} e^x}{\lim_{x\to 1} \frac{1}{2} x^{-1/2}} = \frac{e}{\frac{1}{2}\cdot 1}$$
$$= 2e.$$

3.

$$\begin{aligned}&\lim_{x\to 0^+}\frac{\ln(1+x)}{x^2} \qquad (\text{type } \frac{0}{0})\\ =&\lim_{x\to 0^+}\frac{\frac{1}{1+x}}{2x}\\ =&+\infty.\end{aligned}$$

$u = 1+x$

$$\frac{d\ln(1+x)}{dx} = \frac{d\ln u}{dx} = \frac{d\ln u}{du}\,\frac{du}{dx} = \frac{1}{u}\cdot 1 = \frac{1}{1+x}$$

**Example 6.1.2.** Limits of type $\frac{\infty}{\infty}$

1.

$$\begin{aligned}&\lim_{x\to+\infty}\frac{-x+1}{2x+3} \qquad (\text{type } \frac{\infty}{\infty})\\ =&\lim_{x\to+\infty}\frac{-1}{2}\\ =&-\frac{1}{2}.\end{aligned}$$

*Remark.* The same result can be obtained by dividing both the numerator and the denominator by $x$.

2.

$$\begin{aligned}&\lim_{x\to+\infty}\frac{\ln x}{x^n}, n\in\mathbb{N} \qquad (\text{type } \frac{\infty}{\infty})\\ =&\lim_{x\to+\infty}\frac{\frac{1}{x}}{nx^{n-1}}\\ =&\lim_{x\to+\infty}\frac{1}{nx^n}\\ =&0.\end{aligned}$$

when $n > m > 0$

$x^n \to +\infty$ as $x \to \infty$ "faster" than $x^m$

$$\lim_{x\to+\infty}\frac{x^m}{x^n} = 0$$

*Remark.*

"$\ln x \to +\infty$ as $x \to +\infty$ at a slower rate than any $x^a$"

1. L'Hôpital's rule can NOT be applied for determinate form. For example, $\lim_{x\to 1}\frac{x+1}{x+2} = \frac{2}{3}$, but $\lim_{x\to 1}\frac{(x+1)'}{(x+2)'} = \frac{1}{1} = 1$.
2. If $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ is still $\frac{0}{0}$, $\frac{\infty}{\infty}$, then repeat L'Hôpital's rule.
3. L'Hôpital's rule can be used to justify the previous assertion that as $x\to\infty$, higher degree polynomials "grows faster" than lower degree polynomials; exponential functions grow faster than any polynomials; log functions grow slower than any polynomials.